BLIND IDENTIFICATION OF ARMA SYSTEMS USING THE DISCRETE RANDOM DECREMENT

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ABSTRACT

The Random Decrement method is a computationally simple technique which was initially proposed in the control engineering field for the recovery of impulse responses of systems under operation. This paper demonstrates the advantage of formalising this exotic technique in the context of digital signal processing, thus bringing it to the fore of modern blind identification methods. The discrete Random Decrement is shown to verify a type of Yule-Walker system of equations from which the poles of the system can be deduced. In addition, it is proven that the identification of the zeros (minimum and maximum phase) can be achieved in a linear sense by increasing the number of observations to a least three. The conditions of application of the Random Decrement are relatively broad, and its effectiveness is demonstrated by simulations.

KEYWORDS: Random Decrement, Palm's distribution, blind identification, ARMA model, Yule-Walker equations.

1 INTRODUCTION

The Random Decrement method was first introduced by Henry Cole in 1968 [1] in the context of aerospace structures, as a time domain technique for recovering the impulse response of a system excited by unknown white forces. The technique was later given a mathematical basis by Vandiver *et al* in 1982 [2], who showed that it actually gives an estimate of the autocorrelation function of the system response rather than the impulse response function itself. These results were derived in the continuous time domain, thus leading to intricate equations with limited scope. Moreover, it was believed that blind identification was only possible with a Gaussian excitation.

In this communication, we show the advantage of formalising the issue in the discrete-time domain. By assuming a classical ARMA model for the system impulse response, we end up with a set of equations which are similar to the well-known Normal or Yule-Walker Equations. The only difference is that it involves *first-order conditional statistics* instead of second-order statistics. These equations may be solved for the AR coefficients with any of the current very efficient *ad hoc* algorithms. Next, we show that solving for the MA coefficients leads to a non-linear set of equations which may be difficult to handle in practice. Therefore, we derive a closed-form solution which applies when more than two output observations are available.

2 THE DISCRETE RANDOM DECREMENT

The principle of the *continuous-time* Random Decrement technique is fully detailed for instance in references [1], [2], [3], [4]. We shall directly introduce it here in the *discrete-time* context. Let $\{Y([n]\}$, $n \in \mathbb{Z}$ be a discrete stochastic process and let $\mathcal{C}_p^Y$ be a set of conditions on samples $Y[n]$, $Y[n-1]$,...$Y[n-p]$. We shall typically refer to $\mathcal{C}_p^Y$ as a triggering strategy - e.g. typically $\mathcal{C}_1^Y = \{Y[n-1] \leq u \leq Y[n]\}$ for an upcrossing at level u and time n. Now, let $\{n_i\}_{i \geq 0} = \{n_0 < n_1 < n_2 < ...\}$ be the positive instants (triggering points) where conditions $\mathcal{C}_p^Y$ are satisfied by a specific trajectory of the process $Y[n]$. The idea is to attach a signature $\eta_i[m] = y[n_i + m]$ to each triggering point n_i. Obviously, this defines for a given i a stochastic process $\{\eta_i[m]\}$ for which any trajectory of $Y[n]$ generates a realisation. Under assumption of ergodicity of $\{Y[n]\}$, any such trajectory could serve to form the empirical distribution of the observed signatures $\eta_i[m]$ with respect to i. The principle of the construction is depicted in Fig.(1). From the theory of Palm's distributions [5], it can be demonstrated that, for an increasing observation interval, this empirical distribution converges with probability one to the distribution of an arbitrary signature $\{\eta_i[m]\}$ (independently of i).

The Random Decrement is defined as the conditional expectation:

$$D_Y[m] = E\{\eta_i[m]\} \tag{1}$$

From the above discussion, it is independent of i and it is equal with probability one to the average over all signatures $\eta_i[m]$. Thus, a consistent estimator is simply

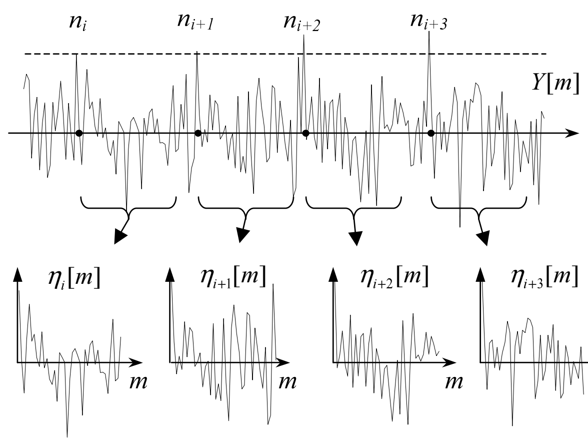

Figure 1: Principle of the Random Decrement construction.

$$\widehat{D}_Y[m] = \frac{1}{I}\sum_{i=0}^{I-1}\eta_i[m] \tag{2}$$

with I the total number of available signatures in the measured signal. In the typical case where $\mathcal{C}_1^Y = \{Y[n-1]\leq u \leq Y[n]\}$, standard but lengthy calculations lead to the following proposition:

Proposition 1: *If $Y[n]$ is a stationary Gaussian process with autocorrelation function $R_Y[m]$, then the Random Decrement $D_Y[m]$ tends to*

$$\sqrt{\frac{F_e}{\pi}}\cdot\frac{R_Y[m-1]-R_Y[m]}{\sqrt{(R_Y[0]-R_Y[1])}} + u\cdot\frac{R_Y[m]}{R_Y[0]} \tag{3}$$

with increasing sampling frequency F_e.

Such a simple form could not be found except in the asymptotic limit, nor for more general triggering strategies. However it could be shown that the Random Decrement only depends on $R_Y[m]$ in general for a Gaussian process. In the next section, it will be shown that the explicit form of the Random Decrement is not required in the blind identification issue.

3 IDENTIFICATION OF ARMA MODELS

Using the above notations, write $\{Y[n]\}, n\in\mathbb{Z}$ as the response of an ARMA system excited by some random stationary zero-mean excitation $\{X[n]\}$:

$$\sum_{i=0}^{2q} a_i Y[n-i] = \sum_{j=0}^{r} b_j X[n-j], \quad a_0 = 1, \quad r\leq 2q \tag{4}$$

Furthermore, depending on the physical nature of $Y[n]$, b_0 will generally be assigned a zero or a non-zero value. Hence, let $\delta_{b_0} = 1$ if $b_0 = 0$ and 0 otherwise. It is easy to verify that application of the Random Decrement to the discrete model (4) leads to

$$\sum_{i=0}^{2q} a_i D_Y[m-i] = \sum_{j=0}^{r} b_j D_X[m-j] \tag{5}$$

3.1 Identification of the poles

For a random white process $\{X[n]\}$, it is easy to show that $D_X[m] = 0$ for $m > -\delta_{b_0} \geq 0$. Therefore, Equation (5) gives

$$\sum_{i=0}^{2q} a_i D_Y[m-i] = 0, \quad m > r-\delta_{b_0} \tag{6}$$

The system of finite-difference Equations (6) is similar to the so-called *Prony's Equations* on the impulse response of a system or the *Yule-Walker Equations* on the autocorrelation function of a system. The only difference is that it involves first-order conditional statistics instead of second-order statistics. It explicitly gives access to the identification of the autoregressive (AR) coefficients $\{a_i\}_{i=0}^{2q}$ - and subsequently to the poles of the system - for example by using any of the very efficient algorithms dedicated to solving the Yule-Walker Equations [6]. Note however that the matrix formed with the $D_Y[m-i]$ may not be toeplitz depending on the choice of the triggering strategy. It is important to realise that the technique holds true under very general conditions, in particular whatever the triggering strategy $\mathcal{C}_p^Y$ and whatever the probability distribution of the excitation process. Finally, in the case where the excitation is not exactly white but has an evanescent autocorrelation function which dies out to zero after some time-lag m_e, Equ. (6) still holds provided r is replaced by $r+m_e$.

3.2 Identification of the zeros

3.2.1 Single-output approach

An approach similar to the previous one is hardly feasible for identifying the zeros since it leads to a non-linear system of equations. Indeed, after equalizing the process $\{Y[n]\}$ by the estimated AR coefficients, the discrete model becomes

$$Z[n] = \sum_{j=0}^{r} b_j X[n-j] \tag{7}$$

Let us now apply the Random Decrement to this finite-difference equation for some given triggering strategy $\mathcal{C}_p^Z$. By observing that $D_X[m] = 0$ for $m < -r-p$ in addition to $m > -\delta_{b_0}$, one gets $2r+p+1-2\delta_{b_0}$ linearly independent equations

$$D_Z[m] = \sum_{j=0}^{r} b_j D_X[m-j], \quad -r-p+\delta_{b_0}\leq m\leq r-\delta_{b_0} \tag{8}$$

In order to solve this system, it is customary to have analytical expressions for the $D_X[m]$. For example, in the ideal case of a Gaussian excitation $\{X[n]\}$ and a simple triggering strategy $\mathcal{C}_0^Z = \{Y[0] = u\}$, it can be shown that

$$D_X[m] = u \cdot \frac{b_m}{\sum_{j=0}^{r} b_j^2}, \quad -r \leq m \leq 0 \tag{9}$$

Obviously, Equations (8) and (9) lead to a difficult non-linear system of equations in the unknown MA coefficients $\{b_j\}_{j=0}^{r}$, a fact that is fully consistent with the second-order statistic case.

3.2.2 Multiple-output approach

One way to turn the estimation of the MA coefficients into a linear problem is to increase the number of measurements positions. The idea follows that of reference [7] on deterministic signals. Consider K measurements $Y_k[n]$, $k = 1, ..., K$ on the system, all of them resulting from the same excitation $X[n]$. After equalising by the estimated AR coefficients, they give $Z_k[n]$, $k = 1, ..., K$ and each measurement is described by a specific set of MA coefficients $\{b_{k,j}\}_{j=0}^{r_k}$. We further assume these coefficients are coprime (no common zeros). Application of the Random Decrement w.r.t. some triggering strategy $\mathcal{C}_p^{Z_c}$ on an arbitrary equalised output $Z_c[n]$ gives K different Random Decrement signatures

$$D_{Z_k}[m] = \sum_{j=0}^{r_k} b_{k,j} D_X[m-j], \quad k = 1, ..., K \tag{10}$$

For simplicity, say $r = \max(r_k, k = 1, ..., K)$ is now the maximum number of MA coefficients. Hence, for any two different observations k_1 and k_2, one can check the following equality[1]:

$$\sum_{l=0}^{r} D_{Z_{k_1}}[m-l]\, b_{k_2,l} = \sum_{l=0}^{r}\sum_{j=0}^{r} b_{k_1,j} D_X[m-l-j]\, b_{k_2,l}$$

$$= \sum_{j=0}^{r} b_{k_1,j} D_{Z_{k_2}}[m-j], \quad -p + \delta_{b_0} \leq m \leq r \tag{11}$$

So for any pair of observations k_1 and k_2, $k_1 \neq k_2$, there are $(r + p + 1 - \delta_{b_0})$ linearly independent equations of the form (11). Taken over all observations, this gives a total of $(r + p - \delta_{b_0} + 1)K(K-1)/2$ equations for solving a maximum of Kr unknown MA coefficients. Because there is a fundamental physical indeterminacy concerning the recovery of the absolute magnitude of the MA coefficients, it is customary to set one of the unknown to an arbitrary value. Therefore, one should use enough sensors so that

$$(r + p - \delta_{b_0} + 1)\frac{K(K-1)}{2} \geq Kr - 1 \tag{12}$$

Condition (12) is satisfied as soon as $K \geq 3$, whatever the values of r, p and δ and therefore in particular if $X[n]$ has some finite memory $m_e \geq 0$.

Proposition 2: *If the Random Decrement is applied on more than two different observations, the coprime MA coefficients of a linear, stable and causal system subjected to a zero-mean stationary excitation can be uniquely recovered within a scaling factor by solving a linear system of equations.*

Finally, the zeros of the system are deduced from the identified MA coefficients. Note that the zeros are *uniquely identified* despite the scaling uncertainty which only affects the MA coefficients. Here again, the identification applies whatever the triggering strategy and whatever the probability distribution of the excitation. Moreover, the excitation does not need to be white. However, a condition for the method is that it requires the exact (maximum) order of the MA parts of each path to be known *a priori*.

4 SIMULATIONS

This section illustrates the proposed approach for identifying a simulated system with 3 eigenmodes. The system was represented by three ARMA(6,6) filters in parallel whose modal parameters are given in the first columns of Tables 1 and 2 (poles were chosen to be global and one zero of the second filter to be maximum phase). It was subjected to a local excitation $X[n]$ synthesised with a 65536 sample-long white Laplacian sequence and the three responses $Y_1[n]$, $Y_2[n]$ and $Y_3[n]$ were measured, as illustrated in Fig.(2). Firstly, the discrete Random Decrement $D_{Y_1}[m]$, $D_{Y_2}[m]$ and $D_{Y_3}[m]$ were estimated by applying the triggering strategy $\mathcal{C}_1^{Y_1} = \{Y_1[1] \geq 0 \geq Y_1[0]\}$. Equation (6) was then used for identifying the poles of the system in conjunction with the Prony algorithm presented in reference [6]. Secondly, based on these estimates, the observations were equalised to give three new processes $Z_1[n]$, $Z_2[n]$ and $Z_3[n]$. New discrete Random Decrement signatures $D_{Z_1}[m]$, $D_{Z_2}[m]$ and $D_{Z_3}[m]$ were estimated by applying the triggering strategy $\mathcal{C}_1^{Z_1} = \{Z_1[1] \geq 0 \geq Z_1[0]\}$. These were finally used to build up a 21×6 system of linear equations from which the MA coefficients could be deduced and ultimately the zeros of the system. Note that in this particular case, the equalisation step was indeed unnecessary since the poles were global. Examples of estimated Random Decrement signatures are displayed in Fig.(3).

The experiments were conducted 100 times, once with an ideal signal to noise power ratio ρ of infinity and once with $\rho = 10$. Results are reported in Table 1 and 2, where the mean values of the estimated poles and zeros (written in terms of normalised frequencies and dampings) are displayed along with their standard deviations. As can be seen, the estimation of the poles was excellent even in the case of additive noise, and that of the

[1] From Equation(8), the first equality holds true only if $-r - p + \delta_{b_0} \leq m - l \leq r - \delta_{b_0}$ with $l = \delta_{b_0}, ..., r$, that is only if $-p + \delta_{b_0} \leq m \leq r$.

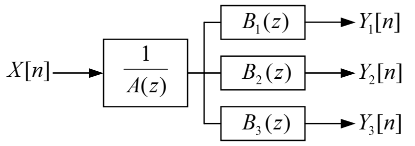

Figure 2: System to identify.

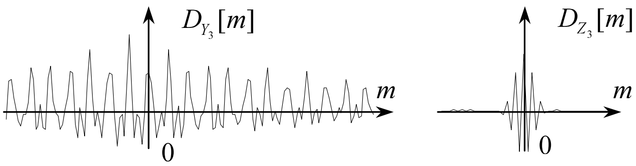

Figure 3: Example of estimated Random Decrement on $Y_3[n]$ and $Z_3[n]$.

zeros as well in the noise-free case (the maximum phase zero, i.e. with negative damping, was perfectly recognised). However when some noise was present, the performances deteriorated more rapidly for the zeros than for the poles, as demonstrated by larger biases and standard deviations.

5 CONCLUSION

This paper demonstrated the advantage of reformulating the Random Decrement - mean value of a Palm's distribution - in the discrete-time context. Not only did it simplify the theoretical formalism of the technique, but it also opened the way to new results. As a matter of fact, it was shown that the Random Decrement applied to an ARMA time series model gives rise to a system of equations similar to the well-known Yule-Walker Equations. The resolution of this system gives access to the blind identification of the AR coefficients from which the poles of the system can be deduced. Next, it was shown that for the MA coefficients to be identified from linear equations (including minimum and maximum phase zeros), it is necessary to increase the number of output observations to at least three. This is an important theoretical result which suggests a two-stage blind identification algorithm. The benefit of the proposed approach is that it holds whatever the triggering strategy and whatever the probability distribution of the excitation. Simulations have supported the robustness of the recovery of the poles and to a lesser extent that of the zeros. However, further work is necessary to investigate the statistical performance of the method for better comparison with the other existing blind-deconvolution techniques. In particular, the effect of additive noise on faulty triggering decisions should be analysed in detail.

Table 1: Identification of the Poles

	Frequencies (normalised)		
	true	*estim.* ρ=∞	*estim.* ρ=10
1	.1250	.1250(±.0009)	.1250(±.0009)
2	.2500	.2500(±.0009)	.2500(±.0009)
3	.3050	.3050(±.0009)	.3050(±.0010)
	Damping (%)		
1	1.27	1.29(±.03)	1.29(±.07)
2	.64	.64(±.04)	.64(±.04)
3	.52	.52(±.03)	.52(±.04)

Table 2: Identification of the Zeros

	Frequencies (normalised)		
	true	*estim.* ρ=∞	*estim.* ρ=10
	1st measurement		
1	.1914	.1914(±.0000)	.1937(±.0058)
2	.2816	.2816(±.0000)	.2912(±.0026)
	2nd measurement		
1	.2500	.2500(±.0000)	.2617(±.0064)
2	.3041	.3041(±.0000)	.3054(±.0039)
	3rd measurement		
1	.2733	.2733(±.0000)	.2851(±.0065)
2	.5000	.5000(±.0000)	.5000(±.0000)
	Damping (%)		
1	20.49	20.49(±0.00)	21.79(±1.77)
2	3.42	3.42(±0.00)	2.18(±1.37)
	2nd measurement		
1	17.18	17.18(±0.00)	14.87(±1.78)
2	-25.03	-25.03(±0.00)	-24.06(±1.12)
	3rd measurement		
1	1.06	1.06(±0.00)	1.48(±1.16)
2	12.19	12.19(±0.00)	12.16(±0.12)

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